Geometry and Arithmetic are Synthetic

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Contents

Kant's claim that geometry and arithmetic are synthetic is the hard part of his doctrine that they are synthetic *a priori*. His argument is admittedly obscure. His explanation of why "7 + 5 = 12" is synthetic (B.15ff)[(1)] needs much more detail before it will convert skeptics. Nevertheless, I believe that Kant was right that geometry and arithmetic are synthetic, and I believe that the argument for this conclusion can be freed from all obscurity. Indeed, I believe that a clear and convincing case for this position emerges from post-Kantian mathematics itself, in particular, from non-Euclidean geometry, Gödel's first incompleteness theorem, and a certain metatheorem about the consistency of non-standard mathematics.

Kant thought that separate arguments were needed to show that geometry and arithmetic are synthetic.[(2)] He makes his case for geometry at B.4, B.16, B.40, and B.64, and for arithmetic at B.15, B.205 (cf. A.103, A only). I will follow him in this practice.

Only minor features of the arguments in this essay are original. I have collected, clarified, and tightened these arguments, and put them in some historical context. My argument that geometry is synthetic was first made by Leonard Nelson. My first argument that arithmetic is synthetic was first made by Irving Copi in 1949. Neither of these arguments has been discussed significantly in the literature. My second argument that arithmetic is synthetic is based on commonplaces in metamathematics, but has not to my knowledge been used before to support Kant.

The first hurdles to the argument are the definitions of "analytic" and "synthetic". I read Kant to offer two independent definitions or criteria of analyticity. The grammatical test is whether the predicate of a judgment is already contained in the concept of the subject, B.10f (cf. B.193). The logical test is whether I can extract the predicate from the subject "in accordance with the principle of contradiction," B.12 (cf. B.15). He elaborates this at B.190-91:

The truth of analytic judgments can always be adequately known in accordance with the principle of contradiction. The reverse of that which as concept is contained and is thought in the knowledge of the object, is always rightly denied. But since the opposite of the concept would contradict the object, the concept itself must necessarily be affirmed of it.

In short, an analytic judgment is one whose negation is a contradiction. As such, it is what logicians call a *logical truth*.[(3)]

A synthetic judgment is any judgment which is not analytic. The other characteristic features of synthetic judgments, such as being ampliative (B.11), are consequences of their non-analyticity.[(4)]

Kant shows no sign of realizing that he has given two criteria of analyticity rather than one, and offers no argument that the two criteria will always agree. In this paper I will use the logical test of analyticity. It is clearer than the grammatical test, easier to apply, and familiar in the areas of mathematics and metamathematics where I intend to use it.

To recap: An analytic judgment is one whose negation is a contradiction (and whose affirmation is not).[(5)] A synthetic judgment is non-analytic, i.e. a judgment whose affirmation and negation are both non-contradictory. I will argue that geometry and arithmetic are synthetic in this sense. I will not argue any further that this is Kant's sense of the term "synthetic".

I would like this paper to make a point about geometry and arithmetic, and also a point about Kant. But if someone produces a strong exegetical argument that Kant did not mean that analytic truths are logical truths, then I will not dispute it, but will be content to make a point only about geometry and arithmetic.[(6)]

Geometry is synthetic

Because Kant wrote the *Critique of Pure Reason* (1781) before the rise of non-Euclidean geometries, and because he declared geometry to be an *a priori* science, he is often accused of codifying the Euclidean geometry of his day as if it were the only possible geometry.[(7)] Hence the advent of non-Euclidean geometry shortly after the *Critique* appeared, and especially Hilbert's proof in 1899 that it is as consistent as Euclidean geometry, seem to falsify Kant's account of geometry. I will argue, on the contrary, that these developments confirmed his view. At least they confirmed his view that geometry is synthetic.

First we should note the possibility that Kant was not at all surprised or "sandbagged" by the development of non-Euclidean geometries. Kant was a friend and correspondent of the Swiss mathematician, Johann Heinrich Lambert (1728-1777), one of the forerunners of non-Euclidean geometry.[(8)]

Despite this, Kant does speak in a way which gives comfort to his critics. In many passages he suggests that the geometry he considers *a priori* is Euclidean. For example, at B.41:

Geometrical propositions are one and all apodeictic, that is, are bound up with the consciousness of their necessity; for instance, that space has only three dimensions.

Or again at B.204:

[Human intuition grounds such axioms as] that between two points only one straight line is possible, or that two straight lines cannot enclose a space, etc.

In these passages he might mean what his critics think he meant: that the only possible geometry is three-dimensional and Euclidean. But at least two other readings are possible, both of which preserve the possibility that mathematical theories of non-Euclidean spaces may be developed without contradiction. The first is that the space of human experience, and the space of the science of physics, are three-dimensional and Euclidean.[(9)] The second is that the human intuition of space depicts or describes a Euclidean space of three dimensions.[(10)] If Kant is talking about the space of physics and experience, then non-Euclidean geometry cannot refute him; only the Einsteinean physics which incorporates a non-Euclidean geometry can refute him.[(11)]

If Kant is talking about human intuition, then perhaps neither non-Euclidean geometry nor Einsteinean physics can refute him.[(12)]

On the last reading, however, Kant might still be refutable. If he is making a point about human intution, or human imagination and visualizability, then it resembles one made earlier by Leibniz: that we cannot mentally picture a space of more than three dimensions. (Leibniz said we cannot picture more than three mutually perpendicular lines.[(13)]) Eva T.H. Brann defends this view explicitly:[(14)]

Any being in the least like us, in whom the capacity to image both its own thoughts and the world confronting it is central, will have an intimate geometry that is Euclidean....In Euclidean geometry alone is similarity, the preservation of shape across variations of size, possible —and similarity is the *sine qua non* of imaging.

As a claim about human imaging capabilities, this might well be accepted even by life-long specialists in non-Euclidean geometry and Einsteinean physics. They can do their work abstractly, representing the *n* dimensions of a non-Euclidean space analytically by the *n* arguments to a function; they need have no skill in visualizing a space of *n* dimensions.

What would falsify Kant on this point is not the advent of consistent non-Euclidean geometries or experimentally confirmed Einsteinean physics, but the claim made by Rudy Rucker that 15 years of diligent practice had enabled him to attain roughly 15 minutes of "direct vision of four-dimensional space."[(15)]

In short, we ought to distinguish (1) the space described by a mathematical theory, from (2) the space of experience and physics, from (3) the space of human intuition —even if under some theories these are intimately related to one another. If Kant's claims were about the space of experience, physics, or human intuition, then he would not deny, even by implication, the consistency of novel geometries, qua axiom systems or mathematical theories,[(16)] although his followers did some damage by assuming that his arguments had precluded just such theories.[(17)]

Kant supports these distinctions when he says, for example, that

there is no contradiction in the concept of a figure which is enclosed within two straight lines....The impossibility arises not from the concept in itself, but...from the conditions of space and of its determination. (B.268; cf. B.271)

On the other hand, it is at least arguable that the space of physics and experience has more than three dimensions, and that human beings can indeed visualize more than three dimensions. But if so, it is not the rise of non-Euclidean geometries that falsifies Kant, but the progress of empirical physics and the mental calisthenics of mathematical adventurers. I offer no argument that Kant can survive these developments.[(18)] I argue only that even if Kant's claims for the primacy of Euclidean geometry are falsified,[(19)] he was still correct that geometry is synthetic.

With this red herring disposed of, we are ready for the argument that non-Euclidean geometry confirmed Kant's view that geometry is synthetic. The argument is simple, but is best presented after a small amount of historical background. Ever since Euclid, mathematicians were perturbed by the parallel postulate, which held that through a point outside a line, exactly one new line could be drawn parallel to the first line. It seemed true, but so "derivative", or so far from elegant simplicity, that it did not deserve its place among the axioms.[(20)] If it could be derived from the other axioms, then it could be removed from the axiom set and made a theorem. Centuries of failure at this task[(21)] led some mathematicians to try another tack. Gerolamo Sacchieri reasoned, correctly, that if the negation of the parallel postulate implied a contradiction, then the postulate would be confirmed by an indirect proof, and all suspicion about it would be answered. In pursuit of his proof, Sacchieri drew a very large number of consequences from the negation of the parallel postulate. His only mistake was to conclude (1733) that a bizarre but consistent consequence was contradictory.[(22)] Early in the 19th century it dawned on Lobachevski, Bolyai, and Gauss that perhaps two consistent geometries could be developed: Euclid's, containing the parallel postulate as an axiom, and a non-Euclidean geometry, containing its negation as an axiom. Late in the century (1899), David Hilbert proved that this was so.

The success —or demonstrated consistency— of non-Euclidean geometry is precisely what confirms Kant. Hilbert proved that both the affirmation and the negation of the parallel postulate led to consistent geometries. But this means that the parallel postulate itself is *not analytic*: its negation does not lead to contradiction. If it were analytic, then its negation would be a contradiction and could not be an axiom in any consistent system; but Hilbert showed that it could be an axiom in a consistent system. Hence, the parallel postulate is not analytic, but synthetic.[(23)]

Later results proved that any *independent* axiom could be replaced by its negation without introducing inconsistency.[(24)] This means that every independent axiom is also non-analytic, or synthetic.

Hilbert proved that each of the original Euclidean axioms, not just the parallel postulate, is independent. So while it may be that comparatively few geometrical theorems are derived with the aid of the parallel postulate or its negation, all of them are derived from at least one independent —hence, synthetic— axiom.

But even if some of Euclid's theorems are logical truths which need none of his axioms as premises, that fact would not weaken Kant's thesis, for he admits that geometry contains some analytic propositions of just this kind (B.16-17).[(25)]

Arithmetic is synthetic

I will offer two independent arguments that arithmetic is synthetic. The first picks up the story of non-Euclidean geometry where we left off, generalizes one of its results, and applies it to arithmetic. The second is based on entirely different considerations.[(26)]

Non-Euclidean geometry is the first case in the history of mathematics in which an axiom of a consistent theory was replaced by its negation without introducing inconsistency. But it was not the last. It inspired playful experimentation in many areas of mathematics. If a consistent "non-standard" geometry could be created by negating a standard geometric axiom, then perhaps a consistent non-standard arithmetic could be created by negating a standard arithmetic axiom. This turned out to be possible, and before long non-standard theories were the subject of serious study in arithmetic, geometry, logic, set theory, and analysis.

Underlying them all is a principle of logic which, despite its importance, has no particular name. Let me call it the *Non-Standardness Theorem*. According to the Non-Standardness Theorem, if a given proposition is not a theorem of a system, then its negation can be added to the system as an axiom without creating inconsistency.[(27)] For example, if the parallel postulate were not an axiom of Euclidean geometry, and if we couldn't derive it from the other axioms (i.e., if it were independent), then it would not be a theorem. Hence under the Non-Standardness Theorem, we could insert the negation of the parallel postulate into the axiom set of Euclidean geometry without creating inconsistency.

If an axiom is *independent* when it cannot be derived from the other axioms, then under the Non-Standardness Theorem every independent axiom is a window onto non-standardness. It is exactly the sort of proposition whose negation can be added to the axiom set (if the affirmation is dropped from it) without introducing inconsistency, creating an unexplored world of consistent, non-standard mathematics.

Independent axioms like the parallel postulate in Euclidean geometry are only special cases of a general type of proposition of increasing interest to 20th century logic: the undecidable proposition. A proposition is undecidable for a given system when neither it nor its negation can be deduced from the axioms of that system. Like independent axioms, undecidable propositions are windows onto non-standardness. Once we know that a proposition is undecidable in a certain formal system, then we know by the Non-Standardness Theorem that either it or its negation can be added to the set of axioms for that system without introducing inconsistency.

Gödel's first incompleteness theorem (1931) proved that every sufficiently powerful, consistent axiom system of arithmetic will contain undecidable propositions. More profoundly, some of these undecidable propositions will be *truths* of arithmetic. His famous formula which says, in one interpretation, "I cannot be proved in system S" —when S is a sufficiently powerful, consistent system of arithmetic— is demonstrably undecidable in system S, and hence true. But in another interpretation the same formula is a statement about natural numbers. It is a truth about numbers, and yet neither it nor its negation can be deduced from the axioms of standard arithmetic. Of course, we could always add axioms to the system in order to plug this hole. But Gödel showed that even an infinite number of supplementary axioms (provided there was still a computable test of axiomhood) would not plug all the holes; a truth about natural numbers — indeed an infinite number of them— would still be undecidable in the enlarged system.

From this quick overview of Gödel's proof, the confirmation of Kant is quickly derived. Assume that arithmetic is consistent. Under Gödel's theorem, there will be undecidable arithmetic truths for any (sufficiently powerful) axiomatization of arithmetic. If arithmetic truths were analytic, then their negations would be contradictions. Hence the undecidable arithmetic truths would have contradictory negations. But they do not have contradictory negations, as shown by the Non-Standardness Theorem. Hence, arithmetic truths are non-analytic, or synthetic.[(28)]

Now we may turn to the second argument for the thesis that arithmetical truths are synthetic. For this argument we will state an arithmetical truth in the notation of modern predicate logic, and then test its analyticity or logical truth.

First we must appreciate what might be called the three levels or grades of truth recognized by predicate logic. Predicate logic formulas may contain constants standing for individual objects. The set of objects which may be assigned to these constants in an interpretation is called the *domain* of the interpretation. If at least one sequence of objects in the domain, when assigned to the sequence of individual constants in a formula, makes the formula come out true, then we say that the formula is *satisfied*. When every sequence of objects in the domain satisfies the formula, then we say the formula is *true for that interpretation*. If the formula is true for every interpretation, then it is *logically valid* or analytic.

Satisfaction is the weakest level of truth. Truth for an interpretation is stronger, and logical validity (or logical truth) is the strongest. A formula is satisfied or "true for some sequence" when it is true for at least one sequence of objects from the domain of an interpretation. A formula is "true for an interpretation" when it is true for all the sequences of objects possible in a given interpretation. A formula is logically valid when it is true for all interpretations. These levels of truth are summarized in the following table.

	true for some sequence, some interpretation	true for all sequences, some interpretation	true for all sequences, all interpretations
satisfaction	*		
truth for an interpretation	*	*	
logical validity	*	*	*

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Since it is easily shown that only logically valid formulas possess the property that their negations are contradictions, then only logically valid formulas are analytic. So the question whether arithmetical truths are analytic has become whether they are logically valid formulas. If they possess only one of the weaker grades of truth, then they are synthetic.

Now let us translate "7 + 5 = 12" into predicate logic notation. The left side of the equation, "7 + 5", contains a function. Let us restate it thus: *add*(7,5), using standard function notation (which is part of predicate logic). Since *add*(7,5) resolves into one number (or object), the expression behaves like an individual constant. It is the sort of thing which may be used as the argument to a first-order predicate.

Equality is a two-place predicate. For clarity let us use the whole word, *Equal*, for this predicate, rather than simply the letter *E*. Now the formula may be expressed thus:

Equal(*add*(7,5),12).

I have put the two arguments to the predicate *Equal* into another pair of brackets for clarity, though it is not necessary.

For our purposes here, "7", "5", and "12" are symbols —numerals, not numbers. They are individual constants which stand for objects from our domain, which is the domain of natural numbers. We cannot know whether *Equal*(*add*(7,5),12) is true or false until we assign values to these currently meaningless symbols.[29] To make this clear, we may replace the numerals with more opaque symbols:

Equal(*add*(a,b),c).

We know that if we assign the sequence of values, <7,5,12>, to that sequence of symbols, then the formula comes out true. This means that the formula is at least *satisfiable*.

The formula is also satisfied by <1,2,3> and by <100,11,111>. So it is satisfied out of our domain more than once. But it is *not* satisfied by <1,2,6> or by <100,200,111>. Hence the formula is not satisfied by all sequences in the domain, and hence is not true for this interpretation. Since the formula is false for at least one interpretation, it cannot be logically valid.

The surprising conclusion is that truths of arithmetic possess only the weakest sort of truth, and are never analytic.

That arithmetic expressions are satisfiable at best, and not logically valid, is common knowledge in metamathematics. It is shown e.g. in the ways in which logical systems must be extended to permit the derivation of arithmetical truths. A formal system of predicate logic with nothing but logically valid axioms can be consistent and complete in the sense that all logically valid formulas of predicate logic will be theorems of the system. All the theorems of such a system will be logically valid —which, while a great strength, proves that it is not yet adequate to capture arithmetic, since arithmetical truths are not logically valid. To extend it so that it does capture arithmetic, we must add axioms which are *not* logically valid. Only then will we be able to derive theorems which are not themselves logically valid.

Conclusion

The received view is that Kant codified the mathematics of his day and was falsified by the development of consistent, non-standard mathematics. I have argued that the reverse is true. Kant radically reinterpreted the mathematics of his day by regarding it as synthetic rather than analytic. His interpretation has been confirmed, not falsified, by the development of consistent, non-standard mathematics. For these consistent, non-standard theories show very clearly that some mathematical truths can be negated without introducing inconsistency, and to possess this property is very likely what Kant meant by a non-analytic or synthetic truth.

It would be quite unreasonable to demand that Kant's argument be as clear and persuasive as this one. That would be to demand that Kant anticipate several of the greatest mathematical results of the two centuries following his death.[30] But if Kant did not anticipate these results, why did he think that geometry and arithmetic were synthetic? It's very unlikely that he could have gleaned what he needed from the non-Euclidean labors of his friend Lambert.

I do not know the answer, but I offer the following thoughts. If Kant had uncritically accepted the mathematics of his day, as reputed, then he would not have regarded it as synthetic. Regarding mathematics as synthetic, then, was both heterodox in his day and (by the arguments here) prescient. If we let these two facts explain each other, then it becomes plausible to suppose that Kant's reason for departing from the contemporaneous understanding of mathematics is closely related to the reason why he was right to do so. I offer the conjecture that Kant knew what he was asserting, but lacked the mathematical tools —such as the consistency of non-Euclidean geometry, the Non-Standardness Theorem, and the distinction between satisfiability and logical truth— to prove what he knew.

Notes

1. Immanuel Kant, *The Critique of Pure Reason*, trans. Norman Kemp Smith, Macmillan, 1929, 1933. I cite the *Critique* by the second ("B") edition numbers, unless a passage occurs only in the first ("A") edition.

2. Friedman, 1992, explores Kant's reasons for this at p. 83.

3. This reading is supported, e.g. by Hector Neri Casteñeda at p. 144, Lewis White Beck (1967) at p. 19, and Gottfried Martin at pp. 18-19.

4. Hence when Kant asserts that the propositions of geometry are synthetic, he is making a claim about the relation of these propositions to contradiction, not to physical space. So we may disregard those objections to Kant's doctrine which interpret "synthetic" in this context as making reference to physical space, e.g. A. J. Ayer (1952) at p. 82. See the quotation from Ayer in footnote 7.

5. On the importance of the parenthetical clause, which strictly speaking goes beyond Kant's text, see Williamson at p. 499.

6. Newton Garver, 1969, distinguishes six definitions or criteria of analyticity in Kant. He considers and rejects what I call the logical test at pp. 255-57. His argument is that Kant's own examples of analytic judgments are not obviously logical truths, but require considerable 'analysis' to reveal their conformity to the criterion of the principle of non-contradiction. He reads Kant's examples correctly; however, he mistakenly assumes that all logical truths will obviously or self-evidently appear to be logical truths. Indeed, he assumes that logical truth and self-evidence are biconditional. But the experience of modern logic, which he cites on his behalf, undermines his assumption that all logical truths are self-evident. For in modern logic, logical truths can be arbitrarily long (finite) strings of symbols, which can require much parsing and analysis before we can prove them to be logically valid. Kant himself undermines the converse assumption that all self-evident truths are logical truths (or analytic). At B.204-05, Kant says that the axioms of intuition must be synthetic *a priori*, "evident", and general. But if this is so, then some "evident" truths are not analytic.

7. Here are some prominent voices in chronological order.

Max Black, 1933, at p. 188: Kant's doctrine of space "has now become obsolete by the discovery of non-Euclidean geometry."

Newman and Kasner, 1940, at p. 118: "Kant delivered an earthly blow [to the nascent non-Euclidean geometries] by laying down his intuitive notions of space which were hardly compatible with either a four-dimensional or a non-Euclidean geometry."

A.J. Ayer, 1946, at p. 82: "The mathematical propositions which one might most pardonably suppose to be synthetic are the propositions of geometry. [If geometry were the study of physical space, then Kant would be right about this.] But while the view that pure geometry is concerned with physical space was plausible enough in Kant's day, when the geometry of Euclid was the only geometry known, the subsequent invention of non-Euclidean geometry has shown it to be mistaken."

Albert Einstein, 1949, at pp. 678-79: Kant's "erroneous opinion" of the character of mathematics was "difficult to avoid in his time," that is, prior to the development of non-Euclidean geometries.

Felix Grayeff, 1951, at p. 171: "Kant's description of organised space assumes that Euclidean geometry is the only possible geometry."

Hans Reichenbach, 1951, at p. 48: That no truths are synthetic *a priori* has become clear "only now, after the physics of Newton and the geometry of Euclid have been superseded."

Stephan Körner, 1955, at p. 26: Kant held that "the presuppositions of arithmetic and Euclidean geometry" and "Newtonian physics" were "absolute synthetic presuppositions of our thinking."

Max Born, 1965, at p. 170: Einstein's special theory of relativity "led...to a new doctrine of space and time. Kant's ideas of space and time as *a priori* forms of intuition were thus finally refuted."

Jonathan Bennett, 1966, at p. 4: Kant's views on geometry, "though ingenious, have been revealed by later work on the philosophy and logic of mathematics as thoroughly and tiresomely wrong."

Morris Kline, 1972, at p. 862: "Kant affirmed, and his contemporaries accepted, that the physical world must be Euclidean....[Kant's view implied] the uniqueness and necessity of Euclidean geometry."

Rudy Rucker, 1977, at p. 21, concludes that Kant's theory of geometry and space constituted an "argument for the truth of the Fifth [Parallel] Postulate [of Euclid]."

8. For details on Lambert's contribution to non-Euclidean geometry, see e.g. Roberto Bonola, 1955, at pp. 44-51. Lambert's most important work on the parallel postulate was published in 1766, long before the *Critique*. I should add that it is not at all clear what sort of preview of non-Euclidean geometry Kant might have learned from Lambert. On the one hand, Humphrey at p. 509 notes that Sacchieri's manuscript "was available" to Lambert. Kant valued his correspondence with Lambert, and for a time considered dedicating the *Critique* to him; see Beck, 1978, at p. 107. On the other hand, in a letter to Kant dated October 13, 1770, Lambert asserted that space has only three dimensions (Zweig, 1967, at p. 64). Martin, 1955, at p. 18, conjectures that Lambert's work helped Kant overcome Leibniz's view that geometry was non-axiomatic or analytic.

9. A.C. Ewing, 1938, at pp. 42f: Kant's theory is compatible with the mathematical success of non-Euclidean geometry, and only rules out its physical applicability.

W.H. Walsh, 1976, at p. 26: Kant implies that "other geometries, though not logically ruled out, would be essentially idle: their conclusions might follow from their premises, but would have no application to the world as it presents itself in experience."

This reading is supported by the fact that Kant once made an explicit argument that Newton's inverse-square law proved that physical space was three-dimensional. See §10 of his 1747 "Thoughts on the True Estimation of Living Forces." Under Kant's influence, Schelling, Hegel, Trendelenberg, and Ueberweg offered similar arguments. See Jammer, 1954, at pp. 174-78.

Norman Kemp Smith favors this reading for the *Critique*, but finds the relevant passages in the *Prolegomena* more flexible; Smith, 1984, at pp. 119-20.

10. Howard DeLong, 1971, supports this interpretation. At p. 40, he suggests that Kant's claim that Euclidean geometry is true *a priori* means, not that it is true of "real space" or that it is the only consistent geometry, but that it is "true of our *a priori* intuitions".

However, DeLong seems to conclude that Kant was wrong even about human intuition. Apparently believing that human intuition is not in fact limited to Euclidean spaces, he concludes that the development of non-Euclidean geometries did in fact falsify Kant's theory (DeLong, p. 58).

11. Of course, this claim has been made, notably by Einstein, Max Born, and Hans Reichenbach; see footnote 7, above. There are several ways to argue against this position, apart from the line of thought I take in the text. To the extent that alternative arguments defend Kant only by defending Euclideanism, even in an Einsteinean world, then my argument in the text suggests that this strategy, even if sound, is unnecessary.

Philip Chapin Jones argues that not even Einsteinean physics refutes Kant. If it turns out that light rays "follow Non-Euclidean tracks", then Euclidean geometry would still be true of Euclidean straight lines, even Euclidean straight lines in our physical universe. Jones, 1946, at p. 142.

Ricardo Gómez interprets the work of Italian mathematician, Eugenio Beltrami, to imply that the concepts of non-Euclidean geometry presuppose that human intuition conforms to Euclidean geometry and hence that non-Euclidean geometry is perfectly compatible with Kant's form of Euclideanism. Gómez at pp. 104, 105, 107.

J. E. Wiredu argues that existing non-Euclidean geometries (unlike some conceivable non-Euclidean geometries) contradict nothing in Euclid, and in that sense contradict nothing in Kant. Wiredu at pp. 8, 27. The same is true of any physics incorporating an existing non-Euclidean geometry. Wiredu at pp. 10, 13, 26.

12. Indeed, if Kant is talking about the nature of human intuition or its role in geometry, then he is saying nothing incompatible with non-Euclidean geometry. But he might in that case be falsified by Descartes' analytic geometry, first published (1637) about 150 years before Kant's *Critique*. By showing the intertranslatability of geometry and algebra, Descartes showed that, for most purposes, we can do geometry without diagrams, spatial intuition, or visualization.

Unlike most other readers of Kant, Ted Humphrey contends that Kant never argued that human spatial intuition is Euclidean. See Humphrey at p. 484; cf. p. 509.

13. Galileo made the same argument earlier, but had not limited the claim to what can be imagined or visualized (1967, original 1632) at p. 14.

14. Eva T.H. Brann, 1991, pp. 603-04; cf 607, 610. At p. 613 Brann admits that *n*-dimensional spaces may be visualizable, when $n > 3$. When she says that we cannot visualize non-Euclidean spaces, she is referring more to such features as (1) that the interior angles of a triangle add up to more, or less, than 180 degrees, (2) that the disparity from 180 increases as the size of the triangle increases, and hence (3) that there is no similarity across varying sizes. Brann herself makes no claim that Kant agrees or disagrees with her thesis. J. Vuillemin, however, argued before Brann that Kant affirms what we might now call Brann's thesis, that human spatial intuitions are limited to a geometry characterized by "the independence of form in relation to magnitude —the postulate distinguishing the geometry of Euclid" (Vuillemin, 1969, at p. 155).

For the view that the space of the visual field is not Euclidean but Riemannian (or elliptical), see e.g. Angell 1974; for the view that it is Lobatchevskian (or hyperbolic), see Grünbaum, 1964, at p. 154 (citing Luneburg and Blank).

15. Rucker, 1984, at pp. 7-8. Unfortunately Rucker does not tell us what mental exercises enabled him to visualize 4-space. He makes a slightly weaker claim in Rucker, 1977, at pp. 21f. In the same earlier work Rucker criticized the mental exercises suggested for this purpose by C. Howard Hinton (see Hinton, 1904). See Rucker, 1977, at pp. 127-28. Again without detail, Rucker says, "There are, in my opinion, better and more direct ways of learning to 'see' 4-D space" (Rucker, 1977, at p. 128).

Jeffrey Weeks, 1985, at p. 195, claims that 4-space is certainly visualizable, and offers exercises for learning to visualize it at pp. 189-214.

If Kant could be construed to be saying that visualizing 4-space is extremely difficult, even 'unnatural', but not impossible, then Rucker's testimony would support his claim, not undermine it.

16. See the quotation from Walsh in footnote 9, above. On the claim that non-Euclidean geometry is logically or mathematically possible even if not physically or intuitively possible, see Gómez at p. 104 and Wiredu at pp. 5ff. Ewing adds that even if Kant is somehow falsified by the development of non-Euclidean geometries, it remains the case that Euclidean geometry is the *a priori* science Kant thought it was; Ewing at p. 41.

17. DeLong, 1971, at pp. 40-41, criticizes Kant for impeding the development of non-Euclidean geometries. At p. 47, he refers to evidence "which shows that Gauss was delayed in announcing his results [of 1824] because of the anticipated reaction of the Kantians." Also see Bonola, 1955, at pp. 64, 121; Struik, 1967, at p. 167; and the quotation from Morris Kline in footnote 7, above.

18. Others have, however. See Martin, 1955, at pp. 18, 207, and Williamson at p. 508, for citations to the work of Nelson, Meinecke, and Natorp, arguing that Kant's doctrine not only permitted, but required the existence of non-Euclidean geometries. Also see Wiredu at p. 6.

19. Humphrey at p. 510 argues persuasively that, even if Kant was wrong, his "preference for Euclidean geometry cannot be considered a function of either mere ignorance or rampant apriorism."

20. If the postulate seems more elegant than derivative, the reason may be that in the text I used the simplest known form of the postulate. In Euclid, the postulate possessed a troublesome complexity:

"Let the following be postulated that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

See Euclid (trans. by T.L. Heath) at p. I.154-55.

21. Bonola, 1955, at pp. 1-21, covers the attempts to derive the parallel postulate from the other Euclidean axioms, from the ancient Greeks to the 17th century. Meschkowski, 1964, at pp. 21-34, covers much the same historical ground, but brings the story up to the 19th century.

22. See e.g. Bonola at pp. 22-44.

23. To my knowledge, this argument was first made by Leonard Nelson, 1965, pp. 158ff, esp. p. 164. It is virtually echoed in Wiredu, pp. 5, 6, 10, Humphrey at p. 508, Brittan at pp. 69-70, and Friedman, 1992, p. 81, though in all three cases with any apparent knowledge of Nelson's work. Brittan does not accept this argument, however, because he does not accept what I have called the "logical test" of analyticity. Friedman seems to support the argument at p. 100. But while Friedman is sympathetic to the argument that the independence of the parallel postulate proves that geometry is synthetic, he concludes that it undermines Kant's claim that one particular geometrical system (the Euclidean) could be known *a priori*; see Friedman at pp. 95, 127.n.52. Friedman offers a different, technical argument that Kant was right that geometry and arithmetic are synthetic at pp. 86-87.

Williamson uses a textual argument at pp. 507-08, and uses Nelson's argument at pp. 511-12 (knowing it is Nelson's). Williamson rejects Nelson's argument at p. 512, claiming that the negation of the parallel postulate conjoined with the other Euclidean axioms may produce no contradiction, but it doesn't follow that the negated parallel postulate taken alone is not contradictory. He believes a contradiction may obtain in a compound statement (such as a set of axioms), but that it might do so in a simple statement or its negation is "in this context...unintelligible" (p. 512).

24. An independent axiom is one which cannot be derived from the other axioms. I offer more detail on these results in the next section, on arithmetic.

25. See Williamson at p. 503.

26. A third argument, which I do not summarize here, may be found in Casteñeda.

27. For a more formal treatment, including full proofs, see Hunter, 1971. Hunter's metatheorem 33.2 is a non-standardness theorem for truth-functional propositional logic. His metatheorem 45.6 is a non-standardness theorem for a first-order theory of predicate logic. Both are given indirect proofs.

28. Irving Copi seems to have been the first to argue that Gödel's first incompleteness theorem implies that arithmetic is synthetic; see Copi, 1949. Atwell R. Turquette criticized Copi's argument in Turquette, 1950, and Copi answered his objections in Copi, 1950.

My argument expands on Copi's by making use of (1) a more precise test of analyticity, (2) the truth of Gödel's undecidable proposition, and not merely its undecidability, and (3) the Non-Standardness theorem.

Howard DeLong agrees with Copi, but with the qualification that it is Gödel's second incompleteness theorem, not his first, which provides an example of a synthetic *a priori* truth (DeLong at p. 222). Despite DeLong's support for Kant's theory of arithmetic, and his close study of Kant's theory of geometry, he never concluded that any propositions of geometry were synthetic.

Both Copi and DeLong argue that Gödel's results prove that arithmetic is synthetic *a priori*, and not merely synthetic.

29. See Casteñeda at p. 151.

30. Bella K. Milmed concluded from a very different set of arguments:

Surprisingly enough, then, it is in the context of the science and mathematics of the twentieth century, and not of the eighteenth, that the fundamental structure of Kant's philosophy is divested of inconsistent and unworkable appendages and takes on its full significance.

See Milmed, 1961, at p. 73.

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Links

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